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Mixing of two microwaves and emission of low-frequency waves from a plasma slab†

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Abstract. The generation of vlf waves as a result of nonlinear interaction of two microwaves with a plasma slab in the presence of a high dc field is investigated theoretically. The reflected and transmitted vlf power comes out to be very high compared with the case for high-frequency waves. The generated vlf power is found to be maximum for an optimum value of the nonlinearity parameter, which is a function of carrier mass, collision frequency and mass of the scatterer. The vlf power is also found to show maxima and minima at various slab thicknesses.

1. Introduction

Harmonic generation and nonlinear mixing of electromagnetic waves in plasmas have been investigated extensively in recent years and found useful for plasma diagnostics (Ginzburg 1960, Kroll *et al.* 1964, Krenz 1965, Stern and Tzoar 1966, Sodha and Kaw 1966, Dienys and Pozhela 1966, Kaw 1968, Kuhn *et al.* 1968, Richter and Bonek 1968, Kaw 1969). Most of the early workers considered nonlinear mixing in the high-frequency region, due to collisions. Ginzburg (1960), however, studied the vlf region in a plasma ($\omega < \delta\nu$; ω is the wave frequency, ν is the collision frequency and $\delta = 2m/M$ where m is the electron mass and M is the mass of the molecule or neutral particle) and showed that, at these frequencies, the time-varying nonlinear phenomena would be more evident than at high frequencies.

Kaw (1969) has recently investigated the vlf second harmonic generation in a simple model semiconductor. Practically, vlf waves are highly damped in the absence of a static magnetic field and their penetration in the semiconductor is very small, therefore high yields of harmonics cannot be expected at such frequencies. However, if two low-frequency waves ($\delta\nu < \omega_{1,2} < \nu$) interact with a plasma, a difference frequency wave is generated. When the difference frequency falls in the vlf region, high yield of vlf power is expected. Turlock and James (1968) have investigated the problem of nonlinear mixing of two microwaves to give rise to a vlf ($|\omega_1 - \omega_2| \ll \omega_1, \omega_2$) wave in a cylindrical plasma column using the elementary approach. They have considered the mutual interaction of fundamental current densities with the magnetic fields of the waves as the only mechanism responsible for mixing. In this paper we have investigated the generation of vlf waves as a result of nonlinear mixing of two microwaves of nearly the same frequencies, ω_1 and ω_2, in a plasma or a semiconducting slab in the presence of a high dc field. In this geometry the mutual interaction of current densities with the magnetic field of the waves does not give rise to appreciable transverse current or electric field. Therefore the collisional mechanism is dominant for the vlf wave generation.

In § 2 the Boltzmann equation for electrons in an electron–neutral particle collision-dominated plasma or an acoustic phonon scattering-dominated semiconductor has been solved by expanding the distribution function in cartesian tensors and

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retaining terms up to second-order tensor $\mathbf{f}_2$. The expressions for the fundamental and difference frequency components of the current density have been obtained in § 3. In § 4 the wave equation has been solved and the expressions for the fields of the difference frequency wave in the reflected and transmitted waves have been obtained.

2. General theory

Let us consider a homogeneous plasma slab with its interfaces at $Z = 0$ and $Z = a$. A dc electric field is applied along the x direction. Two microwaves of nearly the same frequencies, ω_1 and ω_2, are incident normally from the free-space side ($Z < 0$). The electric vectors of the incident waves are in the x direction. Inside the plasma the distribution function of electron velocity $\boldsymbol{v}$ satisfies the Boltzmann transport equation which, on using the cartesian tensor expansion of the distribution function

$$f = f_0 + \boldsymbol{f}_1 \cdot \frac{\boldsymbol{v}}{v} + \mathbf{f}_2 : \frac{\boldsymbol{vv}}{v^2} + \dots,$$

gives the following equations:

$$\frac{\partial f_0}{\partial t} + \tfrac{1}{3} v \, \nabla \cdot \boldsymbol{f}_1 - \frac{e}{3mv^2} \frac{\partial}{\partial v} v^2 (\boldsymbol{E} \cdot \boldsymbol{f}_1) = \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v} v^2 \left(\nu v f_0 + \frac{kT}{m} \nu \frac{\partial f_0}{\partial v} \right) \tag{1}$$

$$\frac{\partial \boldsymbol{f}_1}{\partial t} + v \, \nabla f_0 - \frac{e\boldsymbol{E}}{m} \frac{\partial f_0}{\partial v} + \tfrac{2}{5} v \, \nabla \cdot \mathbf{f}_2 - \frac{2e}{5mv^3} \frac{\partial}{\partial v} (v^3 \boldsymbol{E} \cdot \mathbf{f}_2) - \frac{e\boldsymbol{B} \times \boldsymbol{f}_1}{mc} = -\nu \boldsymbol{f}_1 \tag{2}$$

and

$$\left\{ \frac{\partial \mathbf{f}_2}{\partial t} + v (\nabla \boldsymbol{f}_1 - \tfrac{1}{3} \nabla \cdot \boldsymbol{f}_1 \mathbf{I}) - \frac{ev}{m} \frac{\partial}{\partial v} \left(\frac{\boldsymbol{E} \boldsymbol{f}_1}{v} - \tfrac{1}{3} \frac{\boldsymbol{E} \cdot \boldsymbol{f}_1}{v} \mathbf{I} \right) - \frac{2e\boldsymbol{B} \times \mathbf{f}_2}{mc} \right\}_2 = -\nu \mathbf{f}_2 \tag{3}$$

where the symbols have their usual meaning (Shkarofsky 1968). In equations (1) to (3) the collision term derived by Desloge and Matthysse (1960) for electron–neutral particle collisions has been used. The same collision term is applicable for the scattering of electrons by acoustical phonons, with the only change of the mass of the scatterer M by the effective mass kT/v_s^2 of the phonon (Shockley 1951), where k, T and v_s are the Boltzmann constant, the lattice temperature and the velocity of sound inside the medium respectively. The electric field $\boldsymbol{E}$ appearing in these equations can be written as

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1 \exp(\mathrm{i}\omega_1 t) + \boldsymbol{E}_2 \exp(\mathrm{i}\omega_2 t) + \boldsymbol{E}_{1-2} \exp\{\mathrm{i}(\omega_1 - \omega_2)t\}$$

where $\boldsymbol{E}_{1-2}$ is the generated field. The other harmonics and combination frequency components of the fields will also be present but we are not interested in these components as they lie above the vlf region and have already been investigated by earlier workers; these components are much weaker than the incident fields. The distribution function is then expanded in time harmonics as

$$f_s = f_s^0 + f_s^1 \exp(\mathrm{i}\omega_1 t) + f_s^2 \exp(\mathrm{i}\omega_2 t) + f_s^{1-2} \exp\{\mathrm{i}(\omega_1 - \omega_2)t\}.$$

The time-independent and ω_1, ω_2 frequency components (fundamental) of $\boldsymbol{f}_1$, from equation (2), come out respectively as

$$f_{1x}^0 = \frac{eE_0}{m\nu} \frac{\partial f_0^0}{\partial v} \tag{4}$$

and

$$f_{1x}^{1,2} = \frac{eE_{1,2}(\nu - i\omega_{1,2})}{m(\nu^2 + \omega_{1,2}^2)} \frac{\partial f_0^0}{\partial v} + \frac{eE_0}{m} \frac{(\nu - i\omega_{1,2})}{(\nu^2 + \omega_{1,2}^2)} \frac{\partial f_0^{1,2}}{\partial v}. \tag{5}$$

In deriving the expressions (4) and (5) we have neglected terms of the order of v_{th}^2/v_{ph}^2 and v_d^2/v_{th}^2, where v_{th}, v_d and v_{ph} are the electron thermal, electron drift and wave phase velocities respectively. In an equilibrium plasma v_d^2/v_{th}^2 is always less than the ratio $2m/M$ (Ginzburg 1960).

f_0^0 and $f_0^{1,2}$ appearing in equations (4) and (5) are to be obtained from equation (1) which in these cases leads to the following equations:

$$-\frac{v^2}{3\nu} \frac{\partial^2 f_0^0}{\partial Z^2} - \frac{e}{3mv^2} \frac{\partial}{\partial v} \{v^2(\boldsymbol{E}_0 \cdot \boldsymbol{f}_1^0 + \tfrac{1}{2}\boldsymbol{E}_1 \cdot \boldsymbol{f}_1^{1*} + \tfrac{1}{2}\boldsymbol{E}_2 \cdot \boldsymbol{f}_1^{2*})\} = \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v} \times \left\{v^2\nu\left(v f_0^0 + \frac{kT}{m} \frac{\partial f_0^0}{\partial v}\right)\right\} \tag{6}$$

and

$$i\omega_{1,2} f_0^{1,2} - \frac{m}{M} \frac{1}{v^2} \frac{\partial}{\partial v}\left\{v^2\nu\left(v f_0^{1,2} + \frac{kT}{m} \frac{\partial f_0^{1,2}}{\partial v}\right)\right\} = \frac{e}{3mv^2} \frac{\partial}{\partial v} \{v^2(\boldsymbol{E}_0 \cdot \boldsymbol{f}_1^{1,2} + \boldsymbol{E}_{1,2} \cdot \boldsymbol{f}_1^0)\} \tag{6a}$$

where * denotes the complex conjugate. The first term in equation (6), for high fields, comes out to be of the order of

$$\left(\frac{\delta\omega_1^2}{\nu^2}\right)\left(\frac{v_{th}^2}{v_{ph}^2}\right)\left(\frac{\chi_1^2}{\eta_1^2}\right)\left(\frac{E_1 E_1^* + E_2 E_2^*}{2E_0^2 + E_1 E_1^* + E_2 E_2^*}\right)$$

(where χ_1 and η_1 are the damping coefficient and refractive index corresponding to the fundamental wave) compared with the last term, while it is still smaller for weak fields and hence can be neglected. Further, in the approximation of high dc fields ($E_0^2 \gg E_1 E_1^*, E_2 E_2^*$) the space-independent part of f_0^0 is much greater than the space-dependent part. It is also easy to see from equation (6*a*) that the effect of $f_0^{1,2}$ on $\boldsymbol{f}_1^{1,2}$ comes out to be of the order of $v_d^2\nu/v_{th}^2\omega_{1,2}$ which affects the real part of the refractive index of the fundamental waves by an order of m/M if $\omega_p^2 < \epsilon\nu^2$ (where ω_p is the plasma frequency and ϵ is the lattice dielectric constant) and by an order of $\delta\nu/\omega_{1,2}$ if this inequality does not hold. However, its effect on the imaginary part of the refractive index comes out to be of the order of m/M in both the cases. Therefore the last term in equation (5) can be neglected. Then in the limit $\nu^2 \gg \omega^2$, equation (6), on using equations (4) and (5), has the following solution:

$$f_0^0 \propto \exp\left\{-\int_0^v \frac{mv\,dv}{kT\{1 + e^2M(E_1E_1^* + E_2E_2^* + 2E_0^2)/6m^2kT\nu^2\}}\right\}. \tag{7}$$

For a weakly ionized plasma or acoustic phonon scattering-dominated semiconductor, i.e. $\nu = \nu_0 u$ (where $u = (m/2kT)^{1/2}$—v being the dimensionless electron velocity), this results in

$$f_0^0 = N_0(u^2 + \alpha)^\alpha \exp(-u^2) \tag{7a}$$

where $\alpha = \alpha_0(E_1E_1^* + E_2E_2^* + 2E_0^2)$ is the nonlinearity parameter,

$$\alpha_0 = e^2M/6m^2kT\nu_0^2$$

and N_0 is the normalization constant which is essentially a function of α. In the limit of not too strong microwaves ($E_{1,2} < E_0$) the nonlinearity parameter reduces to $\alpha = 2\alpha_0 E_0^2$. For the particular case of $\alpha \gg 1$, equation (7*a*) reduces to

$$f_0^0 = \left(\frac{m}{2\pi kT}\right)^{3/2} \frac{\sqrt{\pi}(2\alpha)^{-3/4}}{\Gamma(3/4)} \exp\left(-\frac{u^4}{2\alpha}\right) \tag{7b}$$

the well-known Druyvesteyn distribution function (Ginzburg 1960).

For the difference frequency ($\omega_1 - \omega_2$) component of f_1, equation (2), in the limit $|\omega_1 - \omega_2| \ll \nu$, reduces to

$$f_{1x}^{1-2} = \frac{eE_0}{m\nu}\frac{\partial f_0^{1-2}}{\partial v} + \frac{eE_{1-2}}{m\nu}\frac{\partial f_0^0}{\partial v}. \tag{8}$$

The other terms come out to be of the order of $(m/M)\nu/\omega_{1,2}$ times the retained terms, therefore these terms have been neglected. In the evaluation of f_{1x}^{1-2}, we need f_0^{1-2} which from equation (1) is given by

$$\frac{\partial f_0^{1-2}}{\partial u} + 2uf_0^{1-2} + \frac{e^2 E_0^2 M}{3m^2 kT\nu_0^2}\frac{1}{u^2}\frac{\partial f_0^{1-2}}{\partial u} = -\frac{e^2(E_1 E_2^* + 2E_0 E_{1-2})M}{3m^2\nu_0^2 kT}\frac{1}{u^2}\frac{\partial f_0^0}{\partial u}. \tag{9}$$

The solution of this equation in the limit of $E_{1,2}^2 \ll E_0^2$ is

$$f_0^{1-2} = \frac{N_0\alpha(E_1 E_2^* + 2E_0 E_{1-2})}{E_0^2}(u^2+\alpha)^\alpha\left\{\frac{\alpha}{u^2+\alpha} + \ln(u^2+\alpha) - K(\alpha)\right\}\exp(-u^2) \tag{10}$$

where

$$K(\alpha) = \frac{\int_0^\infty u^2(u^2+\alpha)^\alpha\{\alpha/(u^2+\alpha) + \ln(u^2+\alpha)\}\exp(-u^2)\,du}{\int_0^\infty u^2(u^2+\alpha)^\alpha \exp(-u^2)\,du}. \tag{11}$$

The constant of integration of equation (10) has been evaluated by assuming that the time-dependent part of electron density is negligible (Kaw 1969), i.e. $\int_0^\infty v^2 f_0^{1-2}\,dv \simeq 0$. We are justified in assuming this because the space variations of density have already been neglected. Now with this expression (10) for f_0^{1-2}, equation (8) gives the following solution:

$$\begin{aligned} f_{1x}^{1-2} = {} & \frac{eE_0}{m\nu}N_0\alpha\left(\frac{E_1E_2^* + 2E_0E_{1-2}}{E_0^2}\right)\left(\frac{m}{2kT}\right)^{1/2}(u^2+\alpha)^{\alpha-1}\exp(-u^2) \\ & \times 2u^3\left\{\frac{1-\alpha}{u^2+\alpha} - \ln(u^2+\alpha) + K(\alpha)\right\} - \frac{2eE_{1-2}}{m\nu}\left(\frac{m}{2kT}\right)^{1/2} \\ & \times u^3 N_0(u^2+\alpha)^{\alpha-1}\exp(-u^2). \end{aligned} \tag{12}$$

In the special case of $\alpha \gg 1$, f_{1x}^{1-2} comes out to be

$$\begin{aligned} f_{1x}^{1-2} = {} & \frac{eA_0}{m\nu}\exp\left(\frac{-u^4}{2\alpha}\right)\left\{\frac{E_1E_2^* + 2E_0E_{1-2}}{E_0}\left(\frac{7}{2}\frac{u^3}{\alpha} - \frac{u^7}{\alpha^2}\right) - 2E_{1-2}\frac{u^3}{\alpha}\right\} \\ & \times\left(\frac{2kT}{m}\right)^{-1/2} \end{aligned} \tag{12a}$$

where

$$A_0 = \left(\frac{m}{2\pi kT}\right)^{3/2}\frac{\sqrt{\pi}(2\alpha)^{-3/4}}{\Gamma(\frac{3}{4})}.$$

3. Evaluation of current density

The current densities, as defined by

$$J = -\tfrac{4}{3}\pi Ne \int_0^\infty v^3 f_1 \, dv$$

for the fundamentals from equations (5) and for the $(\omega_1 - \omega_2)$ frequency component from equation (12), come out to be

$$J_x^{1,2} = \frac{2\omega_p^2 N_0 E_{1,2}}{3\nu_0^2}\left(\frac{2kT}{m}\right)^{3/2} \int_0^\infty \exp(-u^2)\, u^4 (u^2+\alpha)^{\alpha-1}(\nu_0 u - i\omega_{1,2})\, du \quad (13)$$

and

$$J_x^{1-2} = -\frac{2\omega_p^2 \alpha N_0}{3\nu_0}\frac{(E_1 E_2^* + 2E_0 E_{1-2})}{E_0}\left(\frac{2kT}{m}\right)^{3/2} \int_0^\infty (u^2+\alpha)^{\alpha-1} u^5 \exp(-u^2)$$
$$\times\left\{\frac{1-\alpha}{\alpha+u^2} - \ln(u^2+\alpha) + K(\alpha)\right\} du + \frac{2\omega_p^2 E_{1-2}}{3\nu_0} N_0 \left(\frac{2kT}{m}\right)^{3/2}$$
$$\times \int_0^\infty u^5 \exp(-u^2)\,(u^2+\alpha)^{\alpha-1}\, du \quad (14)$$

where $\omega_p = (4\pi Ne^2/m)^{1/2}$. In the special case $\alpha \gg 1$, the expressions (13) and (14) reduce to

$$J_x^{1,2} = \frac{\omega_p^2}{3\Gamma(3/4)\nu_0^2}(2\alpha)^{-1/2} E_{1,2}\{\nu_0 \Gamma(3/4)(2\alpha)^{1/4} - i\omega_{1,2}\Gamma(5/4)\} \quad (14a)$$

and

$$J_x^{1-2} = -\frac{\omega_p^2}{\nu_0}\frac{(2\alpha)^{-1/4}}{24\sqrt{\pi}\,\Gamma(3/4)}\left(\frac{1}{2}\frac{E_1 E_2^*}{E_0} - E_{1-2}\right). \quad (14b)$$

4. Evaluation of electric intensity

The one-dimensional wave equation is written as

$$\frac{\partial^2 \boldsymbol{E}}{\partial Z^2} + \frac{\omega^2}{c^2}\left(\epsilon \boldsymbol{E} - \frac{4\pi i}{\omega}\boldsymbol{J}\right) = 0. \quad (15)$$

The solution of this equation for the fundamental electric fields inside the plasma, on using equation (13) is

$$E_{1,2} = E_{1,2}^{P}(0)\exp(-ik_{1,2}Z) + E_{1,2}^{R}(0)\exp(ik_{1,2}Z) \quad (16)$$

where $E_{1,2}^{P}$ and $E_{1,2}^{R}$ are the electric intensities of the forward- and backward-propagating fundamental waves inside the plasma, respectively, and

$$k_{1,2}^2 \equiv \frac{\omega_{1,2}^2}{c^2}(\eta_{1,2} - i\chi_{1,2})^2$$
$$= \frac{\omega_{1,2}^2}{c^2}\left\{\epsilon - \frac{8\pi\omega_p^2}{3\nu_0^2} N_0 \left(\frac{2kT}{m}\right)^{3/2} \int_0^\infty \left(1 + \frac{i\nu_0 u}{\omega_{1,2}}\right) u^4 (u^2+\alpha)^{\alpha-1} \exp(-u^2)\, du\right\}. \quad (17)$$

The variation of the damping coefficients with nonlinearity parameter is shown in figure 1. The damping coefficient decreases with increasing nonlinearity parameter.

In order to determine the amplitudes, $E_{1,2}^{P}(0)$ and $E_{1,2}^{R}(0)$ of the waves, the continuity conditions on the tangential component of the electric vector and its

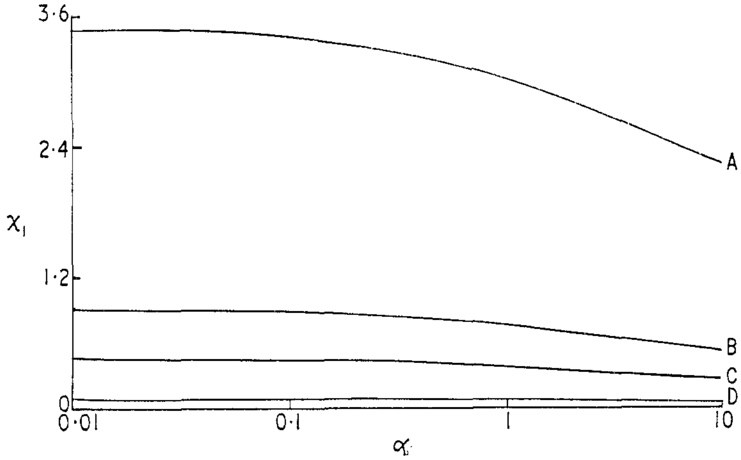

Figure 1. Microwave damping coefficient against α for a germanium slab at 78 K for $\omega_1 = 10^{11}$ rad s^{-1}. Curves A and B correspond to $\nu_0 = 10^{11}$ s^{-1} and $\omega_p^2 = 5\times10^{23}$ and 10^{23} (rad s^{-1})^2 respectively. Curves C and D correspond to $\omega_p^2 = 5\times10^{23}$ (rad s^{-1})^2 and $\nu_0 = 10^{12}$ and 5×10^{12} s^{-1} respectively.

derivative at the boundaries $Z = 0$ and $Z = a$ are to be imposed. Since the fundamental fields outside the plasma slab are

$$E_{1,2}^{i} = E_{1,2}^{i}(0)\exp(-i\omega_{1,2}Z/c) \qquad \text{for } Z < 0$$

$$E_{1,2}^{r} = E_{1,2}^{r}(0)\exp(i\omega_{1,2}Z/c) \qquad \text{for } Z < 0$$

and

$$E_{1,2}^{t} = E_{1,2}^{t}(a)\exp(-i\omega_{1,2}Z/c) \qquad \text{for } Z > a$$

where $E_{1,2}^{i}$ is the wave incident on, and $E_{1,2}^{r}$ and $E_{1,2}^{t}$ are respectively the reflected and transmitted waves from, the plasma slab; the continuity conditions lead to the following expressions for the field amplitudes:

$$E_{1,2}^{P}(0) = A_{1,2}^{P}E_{1,2}^{i}(0) \tag{18}$$

$$E_{1,2}^{R}(0) = A_{1,2}^{R}E_{1,2}^{i}(0) \tag{19}$$

$$E_{1,2}^{r}(0) = (1-\beta_{1,2}^{2})\{\exp(ik_{1,2}a)-\exp(-ik_{1,2}a)\}E_{1,2}^{i}(0)/D_{1,2} \tag{20}$$

and

$$E_{1,2}^{t}(a) = 4\beta_{1,2}\exp(-i\omega_{1,2}a/c)E_{1,2}^{i}(0)/D_{1,2} \tag{21}$$

where

$$\beta_{1,2} = k_{1,2}c/\omega_{1,2}$$

$$D_{1,2} = (1+\beta_{1,2})^2\exp(ik_{1,2}a)-(1-\beta_{1,2})^2\exp(-ik_{1,2}a)$$

$$A_{1,2}^{P} = 2(1+\beta_{1,2})\exp(ik_{1,2}a)/D_{1,2}$$

and

$$A_{1,2}^{R} = -2(1-\beta_{1,2})\exp(-ik_{1,2}a)/D_{1,2}.$$

In the case of the $(\omega_1-\omega_2)$ frequency component of the intensity, equation (15), on using equation (14), reads

$$\frac{\partial^2 E_{1-2}}{\partial Z^2}+k_{1-2}{}^2E_{1-2} = A(E_1{}^{\mathrm{P}}E_2{}^{\mathrm{P}*}+E_1{}^{\mathrm{P}}E_2{}^{\mathrm{R}*}+E_1{}^{\mathrm{R}}E_2{}^{\mathrm{P}*}+E_1{}^{\mathrm{R}}E_2{}^{\mathrm{R}*}) \quad (22)$$

where

$$k_{1-2}{}^2 = \left(\frac{\omega_1-\omega_2}{c}\right)^2(\eta_{\mathrm{d}}-\mathrm{i}\chi_{\mathrm{d}})^2 \quad (23)$$

$$(\eta_{\mathrm{d}}-\mathrm{i}\chi_{\mathrm{d}})^2 = \left[\epsilon+\frac{\mathrm{i}16\pi\omega_{\mathrm{p}}{}^2N_0\alpha}{3\nu_0(\omega_1-\omega_2)}\left(\frac{2kT}{m}\right)^{3/2}\int_0^\infty(u^2+\alpha)^{\alpha-1}u^5\exp(-u^2)\right.$$
$$\times\left\{\frac{1-\alpha}{u^2+\alpha}-\ln(u^2+\alpha)+K(\alpha)\right\}\mathrm{d}u-\frac{\mathrm{i}8\pi\omega_{\mathrm{p}}{}^2N_0}{3\nu_0(\omega_1-\omega_2)}\left(\frac{2kT}{m}\right)^{3/2}\int_0^\infty u^5$$
$$\left.\times\exp(-u^2)(u^2+\alpha)^{\alpha-1}\,\mathrm{d}u\right] \quad (24)$$

and

$$A = -\frac{\mathrm{i}(\omega_1-\omega_2)}{c^2}\left[\frac{8\pi N_0\alpha\omega_{\mathrm{p}}{}^2}{3\nu_0E_0}\left(\frac{2kT}{m}\right)^{3/2}\int_0^\infty(u^2+\alpha)^{\alpha-1}u^5\exp(-u^2)\right.$$
$$\left.\times\left\{\frac{1-\alpha}{u^2+\alpha}-\ln(u^2+\alpha)+K(\alpha)\right\}\right] \quad (25)$$

where η_{d} and χ_{d} are the refractive index and damping coefficient of the $(\omega_1-\omega_2)$ frequency wave. Their variation with α is shown in figure 2 and they decrease more rapidly with α for the difference frequency case than those corresponding to the

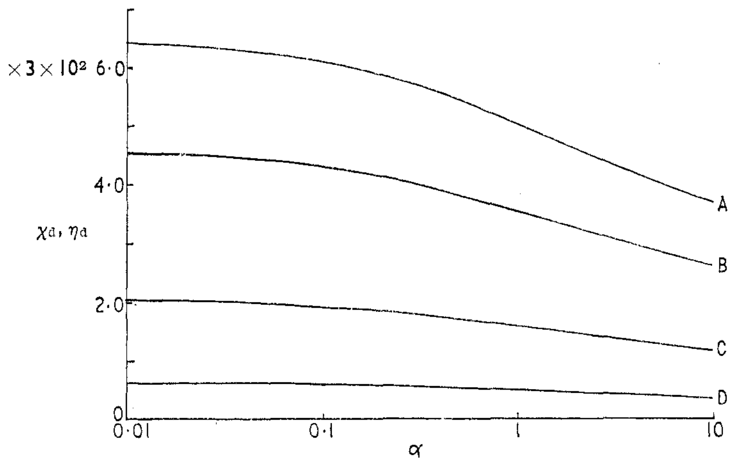

Figure 2. VLF damping coefficient and refractive index against α for a germanium slab at 78 K. Curves A and C correspond to $\nu_0 = 10^{11}\ \mathrm{s}^{-1}$, $\omega_{\mathrm{p}}{}^2 = 10^{23}\ (\mathrm{rad\ s}^{-1})^2$ and $\omega_1-\omega_2 = 10^5$ and $10^6\ \mathrm{rad\ s}^{-1}$ respectively. Curves B and D correspond to $\omega_1-\omega_2 = 10^6\ \mathrm{rad\ s}^{-1}$, $\omega_{\mathrm{p}}{}^2 = 5\times10^{23}\ (\mathrm{rad\ s}^{-1})^2$ and $\nu_0 = 10^{11}$ and $5\times10^{12}\ \mathrm{s}^{-1}$ respectively.

$\omega_{1,2}$ frequencies. For the special case of $\alpha \gg 1$, equations (23) and (25) reduce to

$$k_{1-2}^{2} = \frac{(\omega_1-\omega_2)^2}{c^2}\left\{\epsilon - \frac{\mathrm{i}\omega_p^{2}\sqrt{\pi}(2\alpha)^{-1/4}}{\nu_0(\omega_1-\omega_2)6\Gamma(3/4)}\right\}$$

and

$$A = -\frac{\mathrm{i}(\omega_1-\omega_2)}{c^2}\frac{\sqrt{\pi}(2\alpha)^{-1/4}\omega_p^{2}}{12\nu_0 E_0\Gamma(3/4)}.$$

The general solution of equation (22) is

$$E_{1-2} = E_{1-2}^{\mathrm{P}}(0)\exp(-\mathrm{i}k_{1-2}Z)+E_{1-2}^{\mathrm{R}}(0)\exp(\mathrm{i}k_{1-2}Z) + A\left(\frac{E_1^{\mathrm{P}}E_2^{\mathrm{P}*}+E_1^{\mathrm{R}}E_2^{\mathrm{R}*}}{B_1}+\frac{E_1^{\mathrm{P}}E_2^{\mathrm{R}*}+E_1^{\mathrm{R}}E_2^{\mathrm{P}*}}{B_2}\right) \quad (26)$$

where

$$B_1 = k_{1-2}^{2}-(k_1-k_2^*)^2$$
$$B_2 = k_{1-2}^{2}-(k_1+k_2^*)^2$$

and E_{1-2}^{P} and E_{1-2}^{R} are the forward- and backward-propagating difference frequency wave intensities inside the plasma.

Outside the plasma slab the difference frequency wave intensities are

$$E_{1-2}^{\mathrm{r}} = E_{1-2}^{\mathrm{r}}(0)\exp\{\mathrm{i}(\omega_1-\omega_2)Z/c\} \qquad \text{for } Z<0 \quad (27)$$

and

$$E_{1-2}^{\mathrm{t}} = E_{1-2}^{\mathrm{t}}(a)\exp\{-\mathrm{i}(\omega_1-\omega_2)(Z-a)/c\} \qquad \text{for } Z>a \quad (28)$$

as there is no incident difference frequency wave. The amplitude of the reflected and transmitted waves, on applying the continuity conditions to equations (26), (27) and (28), come out to be:

$$\begin{aligned}E_{1-2}^{\mathrm{r}}(0) = {} & \{(\beta_{1-2}-1)^2\exp(-\mathrm{i}k_{1-2}a)-(\beta_{1-2}+1)^2\exp(\mathrm{i}k_{1-2}a\}^{-1}\\ & \times[\beta_{1-2}(c_1+c_2)\{(\beta_{1-2}-1)\exp(-\mathrm{i}k_{1-2}a)-(\beta_{1-2}+1)\exp(\mathrm{i}k_{1-2}a)\}\\ & +(c_3+c_4)\{(\beta_{1-2}-1)\exp(-\mathrm{i}k_{1-2}a)+(\beta_{1-2}+1)\exp(\mathrm{i}k_{1-2}a)\}\\ & +2\beta_{1-2}(c_5+c_6)-2\beta_{1-2}(c_7+c_8)]\end{aligned} \quad (29)$$

and

$$\begin{aligned}E_{1-2}^{\mathrm{t}}(a) = {} & \exp\{\mathrm{i}(\omega_1-\omega_2)a/c\}\{(\beta_{1-2}-1)^2\exp(-\mathrm{i}k_{1-2}a)-(\beta_{1-2}+1)^2\\ & \times\exp(\mathrm{i}k_{1-2}a)\}^{-1}[2\beta_{1-2}(c_1+c_2)+2\beta_{1-2}(c_3+c_4)-\beta_{1-2}(c_5+c_6)\\ & \times\{(\beta_{1-2}+1)\exp(\mathrm{i}k_{1-2}a)-(\beta_{1-2}-1)\exp(-\mathrm{i}k_{1-2}a)\}\\ & -(c_7+c_8)\{(\beta_{1-2}+1)\exp(\mathrm{i}k_{1-2}a)+(\beta_{1-2}-1)\exp(-\mathrm{i}k_{1-2}a)\}]\end{aligned} \quad (30)$$

where

$$c_5 = \frac{A}{B_1}[A_1^{\mathrm{P}}A_2^{\mathrm{P}*}\exp\{-\mathrm{i}(k_1-k_2^*)a\}+A_1^{\mathrm{P}}A_2^{\mathrm{R}*}\exp\{\mathrm{i}(k_1-k_2^*)a\}]$$

and

$$c_7 = \frac{c(k_1-k_2^*)}{\omega_1-\omega_2}\frac{A}{B_1}[A_1^{\mathrm{P}}A_2^{\mathrm{P}*}\exp\{-\mathrm{i}(k_1-k_2^*)a\}-A_1^{\mathrm{R}}A_2^{\mathrm{R}*}\exp\{\mathrm{i}(k_1-k_2^*)a\}].$$

c_6 and c_8 are obtained from c_5 and c_7, respectively, by interchanging $A_2^{\mathrm{P}*}$ and $A_2^{\mathrm{R}*}$ and replacing k_2^* by $-k_2^*$; c_1, c_2, c_3 and c_4 are obtained from c_5, c_6, c_7 and c_8, respectively, by taking $a = 0$. The expressions (29) and (30) show the complicated effects of interference between forward- and backward-propagating fundamental and difference frequency waves. These effects are obvious from figure 3, where the variation of the difference frequency reflected wave amplitude with slab thickness is plotted for a germanium slab at 78 K.

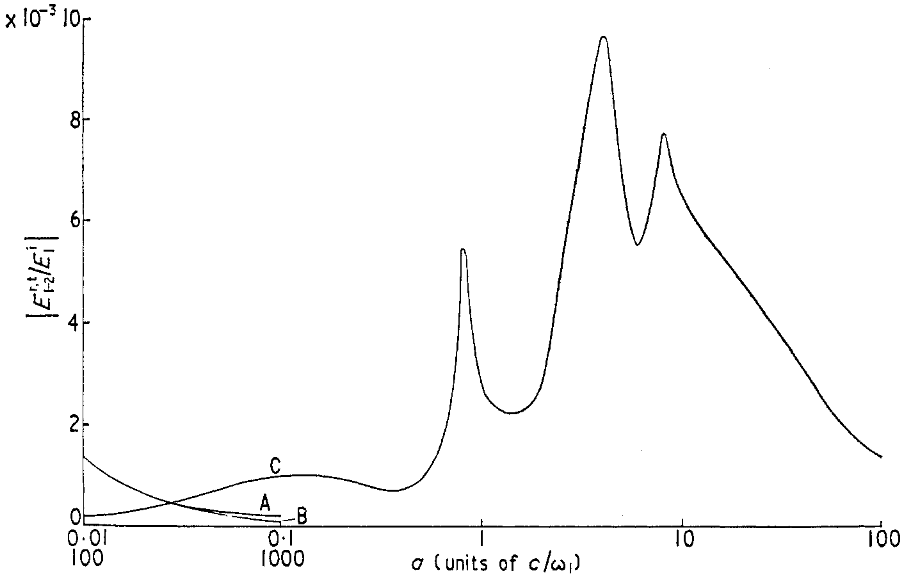

Figure 3. Reflected $E_{1-2}^{r}(0)$ and transmitted $E_{1-2}^{t}(a)$ normalized by $E_1^{i}(0)$ against slab thickness for α = 4 and $E_2^{i}(0)/E_0$ = 0·2. A and B are the continuations of C for $E_{1-2}^{r}(0)$ and $E_{1-2}^{t}(a)$, respectively.

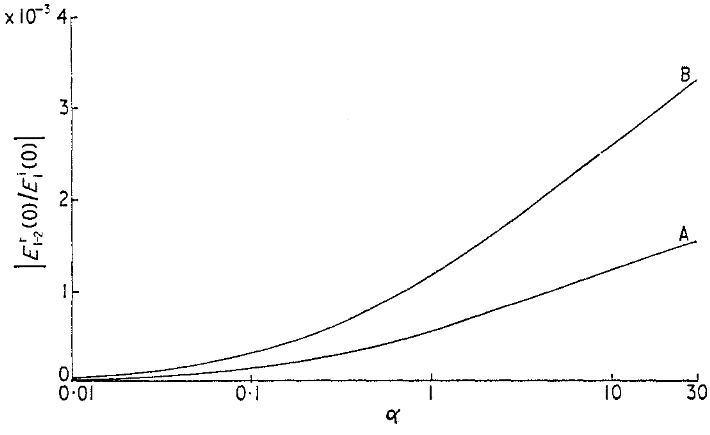

Figure 4. $E_{1-2}^{r}(0)$ normalized by $E_1^{i}(0)$ against α for $E_2^{i}(0)/E_0$ = 0·5, $\omega_p^2 = 10^{23}$ (rad s^{-1})^2 and $(\omega_1-\omega_2) = 10^7$ rad s^{-1}. Curves A and B correspond to $\nu_0 = 10^{12}$ and 5×10^{12} s^{-1} respectively, in the case of a semi-infinite plasma.

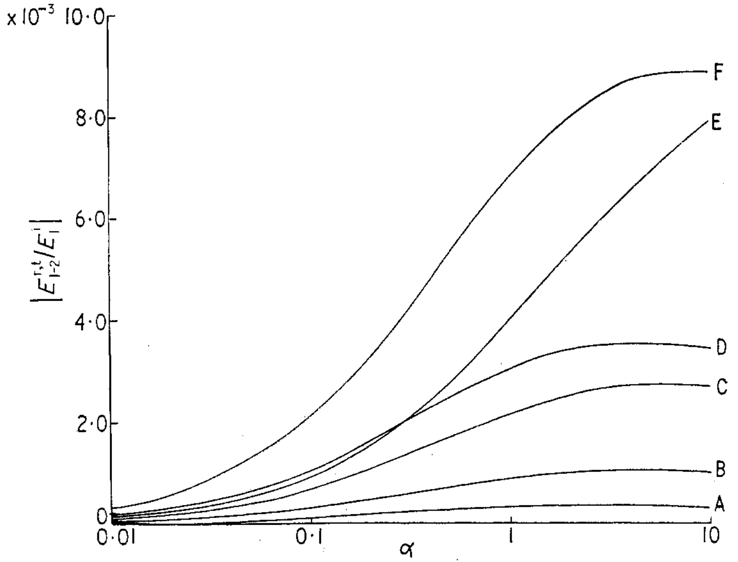

Figure 5. Reflected and transmitted difference frequency wave amplitude, normalized by $E_1^{i}(0)$, against α for $E_2^{i}(0)/E_0 = 0{\cdot}2$, $\omega_1 - \omega_2 = 10^6$ rad s^{-1} and for various slab thicknesses (in units of ω/c) and various collision frequencies. Curves A, D and F correspond to $\nu_0 = 5 \times 10^{12}$ s^{-1} and $a = 0{\cdot}1$, $1{\cdot}0$ and $10{\cdot}0$, respectively. Curves B, C and E correspond to $\nu_0 = 10^{12}$ s^{-1} and $a = 0{\cdot}1$, $1{\cdot}0$ and $10{\cdot}0$ respectively. For curve F the ordinate is to be read after multiplying by 4.

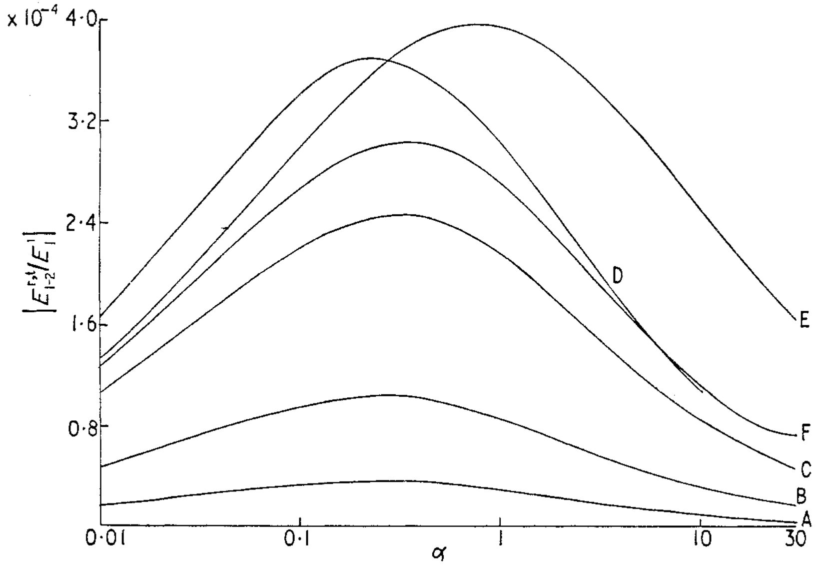

Figure 6. Reflected and transmitted difference frequency wave amplitude, normalized by $E_1^{i}(0)$, against α for $E_2^{i}(0) = 0{\cdot}02\alpha_0^{-1/2}$, $\omega = 10^6$ rad s^{-1} and for various slab thicknesses (in units of ω/c) and various collision frequencies. Curves are numbered as in figure 5 with the same parameters. For curve F the ordinate is to be read after multiplying by 10; ($\omega = \omega_1 - \omega_2$).

In the limit of a semi-infinite medium ($a \to \infty$), the interference effects are only those of the forward-propagating waves and the expression for the reflected wave becomes

$$E_{1-2}^{\mathrm{r}} = \frac{\{4AC/(\omega_1-\omega_2)\}E_1^{\mathrm{i}}E_2^{\mathrm{i}*}}{(k_{1-2}+k_1-k_2^*)\{1+k_{1-2}c/(\omega_1-\omega_2)\}(1+ck_1/\omega_1)(1+ck_2^*/\omega_2)}.$$

The variation of the reflected and transmitted harmonic power with the nonlinearity parameter and collisions has been shown in figures (4) to (6) for a particular case of a germanium slab at 78 K for various collision frequencies. For thin slabs the reflected as well as transmitted power decreases with increasing collisions while the reverse is the case with thicker slabs. This is due to the fact that the difference frequency power is affected by the power absorption of the fundamental waves and by the amplitudes of the backward-propagating waves inside the plasma. The effect of increasing collisions is to decrease the generated difference frequency power by lowering the power absorption on the one hand, and on the other, to raise the generated power by raising the amplitudes of the backward-propagating fundamental waves. The former of the two competing phenomena dominates the latter at small thickness, while the reverse is true for thicker slabs.

5. Conclusions

The damping coefficients of the microwaves as well as the vlf waves decrease with the nonlinearity parameter. In the vlf case the decrease is steeper than in the case of microwaves. The harmonic yield $E_{1-2}^{\mathrm{r,t}}(0)/E_1^{\mathrm{i}}(0)$ is very large at low frequencies, as much as 0·035 for moderately strong dc fields, $E_0 \simeq 0{\cdot}3$ esu, and $E_2^{\mathrm{i}}(0)/E_0 = 0{\cdot}2$. The reflected and transmitted harmonic power has maxima and minima at various slab thicknesses showing the interference effects. On further increasing the slab thickness the amplitude of the reflected power decreases and reaches a saturation point while the amplitude of the transmitted power decreases continuously. The reflected and transmitted harmonic power come out to be maximum for an optimum value of the nonlinearity parameter. This optimum value increases with increasing $(\omega_1-\omega_2)$.

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